Lattice calculation for forward LIBOR model

Tadashi Uratani* Makoto Utsunomiya†

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1 Introduction

We propose a new lattice calculation for Forward LIBOR model in Jamshidian[4], where he shows a difference method based on stochastic differential equation.

Modeling for term structure of interest rates has been shifted from instantenous interest rate (spot rate) model to instantaneous forward rate model such as HJM [2] in direction to utilize for practical derivative pricing. In actual market, there exist no instantaneous rate and most traders use Lognormal model. The typical short term interest rate in the market is 3 month LIBOR(London Interbank Offer Rate) which is used as underlying index for interest sensitive derivatives such as swap, cap and etc. The lognormal model for LIBOR is named "Market Model" by Musiela [1], [5]. Characteristics of forward LIBOR model are choosing the numeraire and changing the probability measure so as to guarantee arbitrage-free and martingale. It makes simple to calculate arbitrage-free prices even for path-depending interest rate sensitive derivative such as knockout caplet.

The rest of paper is organized as follows. Section 1 reviews definitions of bond price process and the induced process of forward LIBOR. Section 2 shows martingale measure to rule out arbitrage among bonds prices. Changing measures make forward LIBOR martingales and we find the relations among those measures. In section 3, we show the lattice algorithm for forward LIBOR in the case of market model. Finally the lattice calculation is introduced for the path-dependent derivative, knockout caplet. We compare the lattice method and other simulation methods of forward and backward induction.

1.1 Discount bond prices and forward LIBOR

We define Discount bond prices and Forward LIBOR. Let $L_i(t)$ be forward LIBOR at time t starting from T_i to T_{i+1} $(0 \leq t \leq T_i)$. When time t equals to the starting time T_i, $L_i(T_i)$ is called a Spot LIBOR. We denote the time interval between T_i and T_{i+1} by δ_i whose unit is a year. For example, when all $L_i(t)$ are 3 month LIBOR, δ_i is 3/12.

In our model, time span is starting from T_0 and ending at T_{N+1}. At T_i, there exists a discount bond which comes maturity date. Let $B_i(t)$ be a discount bond price at time t with maturity date T_i, $\quad i = 1, \cdots, N+1$. Discount bonds are assumed to be default free. The price process is assumed to follow the stochastic differential equation under some regularity assumptions,

$$\frac{dB_i(t)}{B_i(t)} = \mu_i(t)dt + \sigma_i(t)dw(t), \tag{1}$$

*Hosei University,Tokyo
†Bank of Tokyo Mitsubishi

where $w(t)$ is a Wiener process on the probability space $(\Omega, \mathcal{F}, P)$. The forward LIBOR is defined by two bond prices,

$$L_i(t) = \delta_i^{-1}(\frac{B_i(t)}{B_{i+1}(t)} - 1). \tag{2}$$

From definition of bond processes (1) and Ito's lemma, The forward LIBOR process is also Ito's process as,

$$\begin{aligned} dL_i(t) &= \alpha_i(t)dt + \beta_i(t)dw(t), & \text{(3a)} \\ \beta_i(t) &= \delta_i^{-1}\frac{B_i(t)}{B_{i+1}(t)}(\sigma_i(t) - \sigma_{i+1}(t)), & \text{(3b)} \\ \alpha_i(t) &= \delta_i^{-1}\frac{B_i(t)}{B_{i+1}(t)}(\mu_i(t) - \mu_{i+1}(t))) - \beta_i(t)\sigma_{i+1}(t) & \text{(3c)} \end{aligned}$$

On the other hand, a bond price at time t maturing by T_{n+1} is represented by forward LIBORs,

$$B_{n+1}(t) = B_{i(t)}(t) \prod_{i=i(t)}^{n} \frac{1}{1 + \delta_i L_i(t)}$$

where, $i(t)$ is a truncate function matched with time intervals.

2 Forward LIBOR model and arbitrage-free condition

Definitions and theorems necessary for the lattice calculation are outlined according to Jamshidian [3]. The arbitrage-free condition is that there exists a martingale probability measure for relative price process as in theorems. The relative price is defined by the ratio of derivative price divided by a bond price which is called "numeraire".

2.1 Numeraire and the change of probability measure

We choose a numeraire denoting $X(t)$ satisfying positive Ito process

$$\frac{dX(t)}{X(t)} = \mu_X(t)dt + \sigma_X(t)dw(t).$$

$X(t)$ is one of bond prices $B_i(t) \quad (i = 1, \cdots, N+1)$.
Let P^X be the probability measure for the numeraire $X(t)$. If there exists an adapted process $\psi(t)$ such that

$$E_P[\exp(\int_0^{T_{i+1}} \psi(s)dw(s) - \frac{1}{2}\int_0^{T_{i+1}} \psi(s)^2 ds)] = 1.$$

By Girsanov's theorem, we can define a new Wiener process under the equivalent measure P^X with P such as

$$\frac{dP^X}{dP} = \exp(\int_0^{T_{i+1}} \psi(s)dw(s) - \frac{1}{2}\int_0^{T_{i+1}} \psi(s)^2 ds)$$

The new Wiener process is

$$w_X(t) = w(t) + \int_0^t (\phi(s) - \sigma_X(s))ds \tag{4}$$

Where $\psi(t) = \sigma_X(t) - \phi(t)$ satisfies regularity condition.

Theorem 1 [Martingale]
Let $C(t)$ be a derivative price process satisfying

$$\frac{dC(t)}{C(t)} = \mu_C(t)dt + \sigma_C(t)dw(t).$$

Then, $C(t)/X(t)$ is a martingale under the measure P^X if and only if

$$\mu_C(t) = \mu_X(t) + (\sigma_C(t) - \sigma_X(t))\phi(t). \tag{5}$$

Moreover, the derivative price $C(t)$ can be evaluated by the expectation of the measure P^X ,

$$C(t) = X(t)E^X[\frac{C(T)}{X(T)}] \qquad (t \leq T) \tag{6}$$

The process $\phi(t)$ is called the market price of risk for the random process $w(t)$ according to the following Arbitrage-Free condition.

Theorem 2 [Local Arbitrage Free]
Positive process $X(t)$ and$C(t)$ are arbitrage-free if and only if There exists a process $\phi(t)$ satisfying

$$\mu_X(t) - \mu_C(t) = (\sigma_X(t) - \sigma_C(t))\phi(t) \tag{7}$$

In the case of $\sigma_X(t) = 0$ there exists a process $\phi(t)$ such as,

$$\mu_C(t) = r(t) + \sigma_C(t)\phi(t).$$

The condition is well known as local arbitrage-free for one factor model. (7) is a generalized condition for multi-factors.

2.2 Probability measures for forward LIBOR

Substituting (7) of the arbitrage-free condition in theorem 2 to the forward LIBOR process of (3b),

$$dL_n(t) = \beta_n(t)[dw(t) + (\phi(t) - \sigma_{n+1}(t))dt]$$

Under the probability measure for maturity time T_{n+1}, a Wiener process as (4)

$$dw_{n+1} = dw(t) + (\phi(t) - \sigma_{n+1}(t))dt$$

can be defined.Thus,

$$dL_n(t) = \beta_n(t)dw_{n+1}(t).$$

Consequently, forward LIBOR $L_n(t)$ is a martingale under P^{n+1}. Similarly, forward LIBOR L_k for $k = n, \cdots, N$ are satisfying the following stochastic differential equation with initial values at $t = 0$,

$$dL_k(t) = \beta_k(t)dw_{k+1}(t), \tag{8}$$

under probability measure P^{k+1} and forward LIBORs are martingale. We call simply P^{k+1} forward LIBOR measure.

2.3 Relations between forward LIBOR measures

Let $q^k(t, L_i(t); T, L_i(T))$ be the transition probability from $L_i(t)$ at t to $L_i(T)$ at T under P^k . It can be easily shown the proposition.

Proposition 1

The transition probabilities between the measures P^k and P^{k+1} are satisfying ,

$$q^k(t, L_i(t); T, L_i(T)) = \frac{1+\delta_i L_k(T)}{1+\delta_i L_k(t)} q^{k+1}(t, L_i(t); T, L_i(T)). \tag{9}$$

Let $g(t, L_i(t))$ a derivative price of LIBOR depending at time t. By theorem 1,

$$\begin{aligned} E_t^k[g(T, L_i(T))] &= \frac{B_{k+1}(t)}{B_k(t)} E_t^{k+1}[g(T, L_i(T)) \frac{B_{k+1}(T)}{B_k(T)}] \\ &= E_t^{k+1}[g(T, L_i(T)) \frac{1+\delta_k L_k(T)}{1+\delta_k L_k(t)}]. \end{aligned}$$

Thus,the transition probability functions are satisfying

$$q^{N+1}(t, L_i(t); T, L_i(T)) = \prod_{k=n+1}^{N} \frac{1+\delta_i L_k(T)}{1+\delta_i L_k(t)} q^{n+1}(t, L_i(t); T, L_i(T)).$$

3 Lattice algorithm

In LIBOR model, forward LIBOR $L_n(t)$ is function of t which starts from 0 to T_n. When $t = T_n$, $L_n(T_n)$ is called Spot LIBOR which is directly observable in the market. Interest sensitive derivatives depend on the Spot LIBOR. In traditional analysis, Spot LIBOR $L_n(T_n)$ is assumed as a continuous function of T_n $(0 \leq T_n \leq \infty)$. When we price a path dependent derivative, it requires to formulate stopping time modeling. In forward LIBOR model, however we can treat each $L_n(t)$ separately under P^{n+1} and also as a martingale. Binomial trees for Forward LIBOR $L_n(t), \ldots, L_N(t)$ under $P^{n+1}, \ldots, P^{N+1}$ are seen in Figure 1 which is seen how to construct in next sections. The binomial tree of Spot LIBOR $L_n(T_n)$ is shown in the top side of tree triangle of the lattices.

3.1 Lattice calculation for forward LIBOR in Market Model

In Market Model of the volatility $\beta_n(t) = \lambda L_n(t)$ (λ =constant), forward LIBOR process follows geometric Brownian motion. As we have seen in (8), forward LIBOR $L_k(t)$ is a martingale under P^{k+1}. The solution of SDE of(8) is,

$$\begin{aligned} L_k(t) &= L_k(0) \exp\left(-\frac{1}{2}\int_0^t \lambda^2 ds + \int_0^t \lambda dw_{k+1}(s)\right) \\ &= L_k(0) \exp\left(-\frac{1}{2}\lambda^2 t + \lambda w_{k+1}(t)\right) \end{aligned}$$

We use a simple binomial tree for generating forward LIBOR process. Let $0, t_1 = \Delta t, t_2 = 2\Delta t, \ldots, t_m = m\Delta t, \ldots, t$ be the partition from time 0 to time t.

In the time interval Δt , forward LIBOR $L_k(t_m)$ is moved to upside node $L_k^u(t_{m+1})$ or downside node $L_k^d(t_{m+1})$,

$$\begin{aligned} L_k^u(t_{m+1}) &= L_k(t_m) \exp\left(-\tfrac{1}{2}\lambda^2 \Delta t + \lambda\sqrt{\Delta t}\right) \\ L_k^d(t_{m+1}) &= L_k(t_m) \exp\left(-\tfrac{1}{2}\lambda^2 \Delta t - \lambda\sqrt{\Delta t}\right) \end{aligned} \tag{10}$$

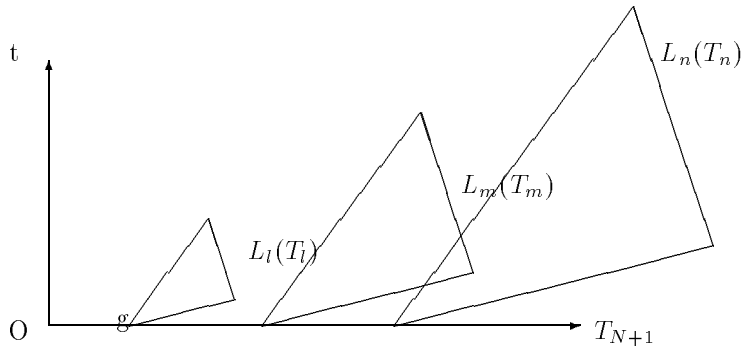

Figure 1: Lattice for forward LIBOR

Let p_u and $p_d(=1-p_u)$ be respectively the transition probability to upside node and downside node.

$$p_u = \frac{\exp\left(\frac{1}{2}\lambda^2\Delta t + \lambda\sqrt{\Delta t}\right) - 1}{\exp\left(2\lambda\sqrt{\Delta}\right) - 1}$$

In lattice tree, let $L_{kj}(t_m)$ be the j-th forward LIBOR node at time t_m. The transition probability from 0 to time t_m is from binomial probability

$$q_j^{k+1}(0, L_k(0), t_m, L_{kj}(t_m)) = \frac{t_m!}{j!(t_m-j)!} p_u^j p_d^{t_m-j}.$$

Firstly we construct all nodes of binomial tree from Forward LIBOR $L_k(0)$ up to time $L_k(T_k)$ under P^{k+1}. In the same way, binomial trees for Forward LIBOR $L_{k+1}(t), \ldots, L_N(t)$ are calculated respectively under $P^{k+2}, \ldots, P^{N+1}$. The problem is, however, that each tree has different probability measure. In order to use the same probability measure, we take P^{N+1}. It is known from (9)

$$q_j^{N+1}(0, L_k(0), t_m, L_{kj}(t_m)) = \frac{t_m!}{j!(t_m-j)!} p_u^j p_d^{t_m-j} \prod_{i=k+1}^{N} \frac{1+\delta_i L_i(0)}{1+\delta_i L_{ij}(t_m)} \quad (11)$$

Let $g(t_m, L_k(t_m))$ be the payoff at t_m in European derivative of Forward LIBOR $L_k(t_m)$. The price of derivative is given from (6)

$$g(0, L_k(0)) = B_{N+1}(0) E_t^{N+1}\left[\frac{g(t_m, L_k(t_m))}{B_{N+1}(t_m)}\right].$$

using the probability of (11)

$$g(0, L_k(0)) = B_{N+1}(0) \sum_j^{A_{t_m}} \left[\frac{g(t_m, L_{kj}(t_m)) \frac{t_m!}{j!(t_m-j)!} p_u^j p_d^{t_m-j}}{B_{N+1}(t_m)} \times \prod_{i=k+1}^{N} \frac{1+\delta_i L_i(0)}{1+\delta_i L_{ij}(t_m)}\right]$$

where let A_{t_m} be the number of nodes at time t_m.

4 Valuation of Knockout Caplet

Caplet is an option contract where option holders receive at time T_{n+1} the difference of rates only if spot LIBOR at T_n exceeds the strike rate K. Let $C(T_n)$ be the value of payoff at time T_n

$$C(T_n) = \delta_n B_{n+1}(T_n) \max(L_n(T_n) - K, 0).$$

Knockout caplet is a caplet where the contract becomes worthless once the spot LIBOR gets below the knockout rate K_o before maturity date. Let $Kpl(T_n)$ be the value of Knockout caplet at time T_n as

$$Kpl(T_n) = B_{n+1}(T_n)\delta_n \max(L_n(T_n) - K, 0) \times I_{\min(L_1(T_1), \ldots, L_{n-1}(T_{n-1})) > K_o}. \quad (12)$$

Let us compare calculation methods for knockdown caplet. Firstly, the price $Kpl(t)$ can get in expectation form under P^{n+1} by,

$$Kpl(t) = B_{n+1}(t)E_t^{n+1}\left[\delta_n \max(L_n(T_n) - K, 0) \times I_{\min(L_1(T_1),\ldots,L_{n-1}(T_{n-1}))>K_o}\right] \tag{13}$$

Forward LIBOR process under measure $P^{n+1}$is

$$dL_m(t) = -\sum_{i=m+1}^{n} \frac{\delta_i \beta_i(t)\beta_m(t)^\top}{1+\delta_i L_i(t)} dt + \beta_m(t) dw_{n+1}(t). \quad (t \le T_m \le T_n)$$

From simulation of this SDE we generate all nodes of LIBOR values backwardly and then calculate the option price. This method is called Backward Induction in Jamshidian [4].

Secondly, under spot LIBOR measure P^B which set numeraire as rolling spot LIBOR price $B(t)$, the option price is

$$Kpl(t) = B_{n(t)}(t)E_t^B\left[\frac{\delta_n \max(L_n(T_n) - K, 0)}{\prod_{j=n(t)}^{n}(1+\delta_j L_j(T_j))} \times I_{\min(L_1(T_1),\ldots,L_{n-1}(T_{n-1}))>K_o}\right]$$

Forward LIBOR process under the measure P^B are satisfying the following SDE and the calculation of price is called forward Induction.

$$dL_n(t) = \sum_{i=n(t)}^{n} \frac{\delta_i \beta_i(t)\beta_n(t)^\top}{1+\delta_i L_i(t)} dt + \beta_n(t) dw_B(t)$$

Thirdly, we propose a lattice algorithm for knockout caplet;
(i)generate binomial trees for each forward LIBOR $L_i(t)$ under the martingale measure P^{i+1} by (10) and at time T_i calculating Spot LIBOR $L_i(T_i)(i = 1,\ldots,n-1)$.
(ii) assign the transition probabilities of same measure P^{n+1} by (11) to all nodes of all trees.
(iii)calculate the probability

$$P^{n+1}[\min(L_1(T_1),\cdots,L_m(T_m)) > K_o]$$

that is all Spot LIBOR $L_i(T_i)$ are larger than knockout rate K_o in the measure P^{N+1}.
(iv) calculate the expectation of $\max(L_n(T_n - K, 0))$ from (12).
We show the comparison of three methods for knockout caplet in the our assumption;

[0]Rolling spot LIBOR price

$$B(t) \equiv \frac{B_{n(t)}(t)}{B_1(0)} \prod_{i=1}^{n(t)-1}(1+\delta_i L_i(T_i))$$

Knockout Caplet Assumption

- Knockout Caplet:3 monthLIBOR:price at time 0 for maturity 3 year
- knockout strike rate:$K_o = 0.005, 0.01, 0.015, 0.02$
- caplet strike rate:$K = 0.02, 0.03$
- log volatility:$\lambda = 0.2, 0.3, 0.4, 0.5$
- LIBOR period:10equal span
- Terminal measure:P^{16+1}
- initial yield:0.03(constant)
- average of simulation times(number of run:10000)
- random number:rnorm function(Splus Version3.2J Release 1 for Sun SPARC)

The comparison of three methods is give in the Table 1. Forward and backward inductions are methods originated in Jamshidian [3]. Lattice is our method in which the high knockout rate gives low caplet price due to proper evaluation knockout probability. The lattice method takes computing time only for generating lattice. however induction methods are simulation to require so many runs to have higher precision.

Table 1: Knockout Caplet $K = 0.02$ Simulation number:10000

		K_o=0.005	0.01	0.015
λ =0.2	Backward	0.002412289	0.002399882	0.002399109
	Forward	0.002433752	0.002434543	0.002419707
	Lattice	0.002406534	0.002402772	0.002234430
0.3	Backward	0.002638880	0.002621019	0.002651906
	Forward	0.002657684	0.002621897	0.002641743
	Lattice	0.002642767	0.002457840	0.001513274
0.4	Backward	0.002953369	0.002959331	0.002875769
	Forward	0.003013726	0.002955638	0.002843283
	Lattice	0.002877610	0.002015249	0.000690205
0.5	Backward	0.003318795	0.003321196	0.003095759
	Forward	0.003142553	0.003214522	0.003163378
	lattice	0.002883561	0.001252366	0.000304366

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